The Nucleon-Nucleon Interaction

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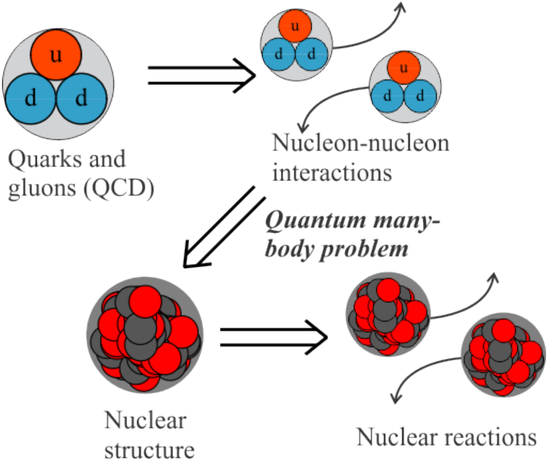
Outline

Introduction

Introduction

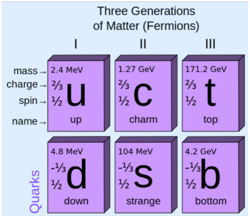


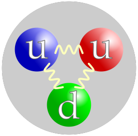
Introduction

- The modern description of the strong interaction of quarks is quantum chromodynamics (*QCD*). This is a gauge theory with a $\mathrm{SU}(3)_c$ gauge group. The strong force is mediated by gauge bosons known as gluons. This gauge symmetry is exact, and the gluons are massless.

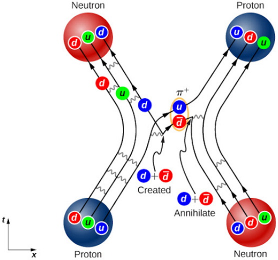
- There are totally six types, a.k.a. flavors of quarks.
- In QCD, each flavor of quark comes in three "copies" of different colour. It is conventional to call these colours red, green and blue, even though they have nothing to do with actual colours. For a flavour f, we can write these as q_f^{red} , q_f^{green} and q_f^{blue} . We can put these into an triplet:

$$q_f = \begin{pmatrix} q_f^{red} \\ q_f^{green} \\ q_f^{blue} \end{pmatrix}$$





Introduction



- The nucleon-nucleon (NN) interaction is a residual interaction of strong interaction described by the QCD.
- The QCD lagrangian density:

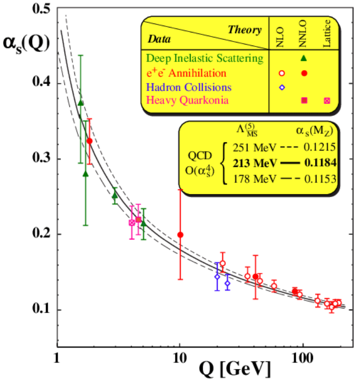
$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \sum_{f=1}^{6} \bar{q}^i_f \left(i\gamma^\mu D^{ij}_\mu - m_f \delta_{ij} \right) q^j_f$$

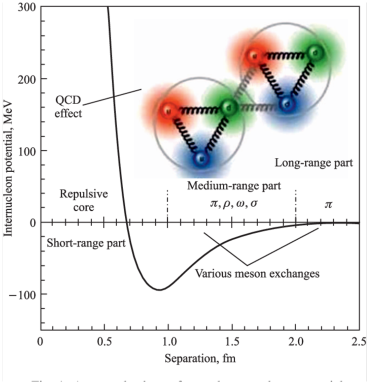
with

$$G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0 f_{abc} A^b_\mu A^c_\nu$$
$$D_\mu \equiv \partial_\mu - igA^a_\mu T_a$$

where A^a_ν are the gluon fields ($a = 1, \ldots, 8$); q^i_f is the quark field with the color index ($i = 1, 2, 3$) and flavor index (f) ; $T_a = \lambda_a/2$ are color SU(3) generators (c.f. Appendix).

Introduction





- The QCD is non-perturbative at low-energy region ($\alpha_s = g^2/4\pi$ increases with the decrease of energy).
- The NN interaction is phenomenologically described in terms of exchange bosons ($\pi, \sigma, \omega, \rho, \cdots$)

General properties

General properties

The general form of NN interaction,

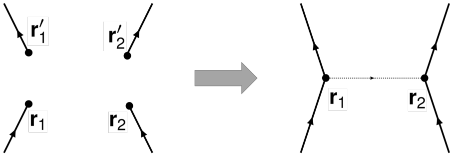
$$\left\langle \mathbf{r}_1' s_1' t_1' \mathbf{r}_2' s_2' t_2' | \widehat{V} | \mathbf{r}_1 s_1 t_1 \mathbf{r}_2 s_2 t_2 \right\rangle$$

where $s_i = \pm 1/2$ and $t_i = \pm 1/2$ are spin and isospin projections. The bras and kets span the product spaces of the coordinate wave functions and the spin and isospin vector, so this is a sufficient basis (since it is complete). Suppressing spin and isospin for the moment, the action of $\widehat{V}$ on the coordinate basis is

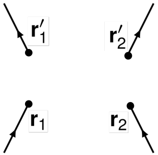
$$\widehat{V} |\mathbf{r}_1 \mathbf{r}_2\rangle = \int V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1' \mathbf{r}_2'\rangle d^3 r_1' d^3 r_2'$$

- The familiar local potential corresponds to the special form

$$V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}_1, \mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{r}_1') \delta(\mathbf{r}_2 - \mathbf{r}_2') \quad \Longrightarrow \quad \widehat{V} |\mathbf{r}_1 \mathbf{r}_2\rangle = V(\mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1 \mathbf{r}_2\rangle$$



General properties



- Taylor expansion of the general potential

$$\left|\mathbf{r}'_1\mathbf{r}'_2\right\rangle = |\mathbf{r}_1\mathbf{r}_2\rangle + \left[\left(\mathbf{r}'_1 - \mathbf{r}_1\right)\cdot\boldsymbol{\nabla}_1 + \left(\mathbf{r}'_2 - \mathbf{r}_2\right)\cdot\boldsymbol{\nabla}_2\right]|\mathbf{r}_1\mathbf{r}_2\rangle + \cdots$$
$$=: \exp\left\{\left(\mathbf{r}'_1 - \mathbf{r}_1\right)\cdot\boldsymbol{\nabla}_1 + \left(\mathbf{r}'_2 - \mathbf{r}_2\right)\cdot\boldsymbol{\nabla}_2\right]\right\} : |\mathbf{r}_1\mathbf{r}_2\rangle$$

where the "normal-ordering" notation $:\widehat{O}:$ means here that the derivatives be moved to act only to the right of the coordinates (and not on the coordinates).

$$\widehat{V}|\mathbf{r}_1\mathbf{r}_2\rangle = \int V\left(\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{r}_1,\mathbf{r}_2\right)\exp\left\{\frac{i}{\hbar}\left(\mathbf{r}'_1 - \mathbf{r}_1\right)\cdot\mathbf{p}_1 + \frac{i}{\hbar}\left(\mathbf{r}'_2 - \mathbf{r}_2\right)\cdot\mathbf{p}_2\right\}|\mathbf{r}_1\mathbf{r}_2\rangle\, d^3r'_1 d^3r'_2$$
$$= \widetilde{V}\left(\mathbf{r}_1,\mathbf{r}_2,\mathbf{p}_1,\mathbf{p}_2\right)|\mathbf{r}_1\mathbf{r}_2\rangle$$

The above general NN potential should preserve some symmetries.

General properties

Considering the NN potential depending on the positions, momenta, spins, and isospins of the two nucleons concerned:

$$V(1,2) = v\left(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{p}_1, \boldsymbol{p}_2, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2\right)$$

- Translational invariance: the dependence on the positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ should only be through the relative distance $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.
- Galilei invariance: the interaction potential should be independent of any transformation to another inertial frame of reference. This demands that the interaction should depend only on the relative momentum $\boldsymbol{p} = \boldsymbol{p}_1 - \boldsymbol{p}_2$.
- Rotational invariance: all terms in the potential should be constructed to have a total angular momentum of zero.

General properties

- Isospin invariance: the only terms that are scalar under rotation in isospin space are those containing: no isospin dependence, the scalar product $\hat{\tau}_1 \cdot \hat{\tau}_2$, or powers thereof.
  With the properties of Pauli matrices,
  $$\left[\hat{\tau}_i, \hat{\tau}_j\right] = 2\mathrm{i} \sum_k \varepsilon_{ijk} \hat{\tau}_k \quad , \quad \left\{\hat{\tau}_i, \hat{\tau}_j\right\} = 2\delta_{ij}, \quad \hat{\tau}_i \hat{\tau}_j = \delta_{ij} + i\epsilon_{ijk}\hat{\tau}_i$$
  one finds all powers of $\hat{\tau}_1 \cdot \hat{\tau}_2$ can be reduced to the first-order product,
  $$(\hat{\tau}_1 \cdot \hat{\tau}_2)^2 = \sum_{ij} \hat{\tau}_{1,i}\hat{\tau}_{2,i}\hat{\tau}_{1,j}\hat{\tau}_{2,j} = 3 - 2\sum_k \hat{\tau}_{1,k}\hat{\tau}_{2,k} = 3 - 2\hat{\tau}_1 \cdot \hat{\tau}_2$$
- Parity invariance: the requirement for the potential is
  $$V(\boldsymbol{r}, \boldsymbol{p}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = V(-\boldsymbol{r}, -\boldsymbol{p}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$$
  containing an even power of $\boldsymbol{r}$ and $\boldsymbol{p}$ together.
- Time reversal invariance: it requires
  $$V(\boldsymbol{r}, \boldsymbol{p}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = V(\boldsymbol{r}, -\boldsymbol{p}, -\hat{\boldsymbol{\sigma}}_1, -\hat{\boldsymbol{\sigma}}_2, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$$
  so that an even number of $p\,\mathrm{s}$ and $\hat{\sigma}\mathrm{s}$ combined are allowed in each term.

Functional form

Functional form

The above constraints lead to

$$V_{NN} = V_1\left(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\right) + V_\tau\left(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\right) \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$$

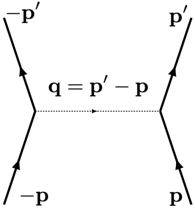
- central parts:
$$V_1(\mathbf{r}, \mathbf{p}) + V_\sigma(\mathbf{r}, \mathbf{p}) \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$$
- vector parts (spin-orbit interaction):
$$V_{LS}(\mathbf{r}, \mathbf{p}) \boldsymbol{L} \cdot \boldsymbol{S},$$
where $\boldsymbol{L} \cdot \boldsymbol{S} = -i\hbar(\mathbf{r} \times \mathbf{p}) \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$.
- tensor parts:
$$V_T(\mathbf{r}, \mathbf{p}) S_{12}(\widehat{\mathbf{r}})$$
with tensor operator in coordinate space
$$S_{12}(\hat{\mathbf{r}}) \equiv \left[3 \frac{(\boldsymbol{r} \cdot \hat{\boldsymbol{\sigma}}_1)(\boldsymbol{r} \cdot \hat{\boldsymbol{\sigma}}_2)}{r^2} - \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2\right] = 3(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\sigma}}_1)(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\sigma}}_2) - \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2$$
where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$.

Functional form

- The tensor force was found to be necessary to explain the properties of the deuteron. It contains the term $(\boldsymbol{r}\cdot\hat{\boldsymbol{\sigma}}_1)\,(\boldsymbol{r}\cdot\hat{\boldsymbol{\sigma}}_2)$, but in such a combination that the average over the angles vanishes. The full expression is

$$S_{12} = \left[v_0(r) + v_1(r)\hat{\tau}_1\cdot\hat{\tau}_2\right] S^{r}_{12}$$

Functional form



$$\mathbf{k} = \frac{1}{2}(\mathbf{p}' + \mathbf{p})$$

Center-of-Mass Frame

The full operator form in the center-of-mass frame:

- in coordinate space:

$$\left\{1_{\rm spin}, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,\ \mathrm{S}_{12}(\widehat{\mathbf{r}}), \mathrm{S}_{12}(\widehat{\mathbf{p}}), \mathrm{L}\cdot\mathrm{S}, (\mathrm{L}\cdot\mathrm{S})^2\right\} \times \left\{\mathbf{1}_{\rm isospin}, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2\right\}$$

  times scalar operator-like functions of r^2, p^2, and L^2 (rather than $\mathbf{r}\cdot\mathbf{p}$).

- In momentum space:

$$\left\{1_{\rm spin}, \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,\ \mathrm{S}_{12}(\widehat{\mathbf{q}}), \mathrm{S}_{12}(\widehat{\mathbf{k}}), \mathrm{i}\mathbf{S}\cdot(\mathbf{q}\times\mathbf{k}), \boldsymbol{\sigma}_1\cdot(\mathbf{q}\times\mathbf{k})\sigma_{\mathbf{2}}\cdot(\mathbf{q}\times\mathbf{k})\right\}$$

  here $\mathbf{q} \equiv \mathbf{p}' - \mathbf{p}$ and $\mathbf{k} \equiv (\mathbf{p}' + \mathbf{p})/2$, times scalar functions of $p^2 \cdot p^2 \cdot$ and $\mathbf{p}\cdot\mathbf{p}'$.

Functional form

The NN potential takes the general form

$$v = v_0(r) + v_\sigma(r)\hat{\boldsymbol{\sigma}}_1 \cdot \hat{\sigma}_2 + v_\tau(r)\hat{\tau}_1 \cdot \hat{\tau}_2 + v_{\sigma\tau}\left(\hat{\boldsymbol{\sigma}}_1 \cdot \hat{\sigma}_2\right)\left(\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2\right)$$

or in the traditional formulation using exchange operators $\hat{P}$:

$$v = v_{\mathrm{W}}(r) + v_{\mathrm{M}}\hat{P}_r + v_{\mathrm{B}}\hat{P}_\sigma + v_{\mathrm{H}}\hat{P}_r\hat{P}_\sigma$$

The indices stand for Wigner, Majorana, Bartlett, and Heisenberg.

- The spin exchange operator $\hat{P}_\tau$:

$$\hat{P}_\sigma = \frac{1}{2}(1 + \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2) = \begin{cases} -1 & \text{for the singlet} \\ +1 & \text{for the triplet} \end{cases}$$

Note that $\hat{P}_r\hat{P}_\sigma\hat{P}_\tau = -1$.

Functional form

- The isospin projection operator $\hat{P}_T$:

$$\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2 = 4\hat{\boldsymbol{t}}_1 \cdot \hat{\boldsymbol{t}}_2 = 2\left[\left(\hat{\boldsymbol{t}}_1 + \hat{\boldsymbol{t}}_2\right)^2 - \hat{\boldsymbol{t}}_1^2 - \hat{\boldsymbol{t}}_2^2\right]$$

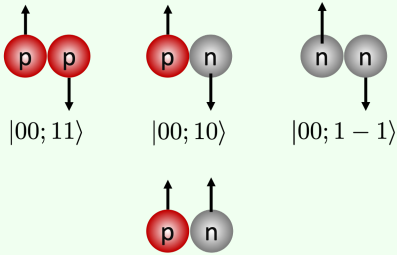
$$= 2\left[T(T+1) - \frac{3}{4} - \frac{3}{4}\right] = \begin{cases} -3 & \text{for the singlet} \\ +1 & \text{for the triplet} \end{cases}$$

This result allows the construction of projection operators onto the singlet or triplet, respectively, which are simply such linear combinations that they yield zero when applied to one of the two states and 1 when applied to the other:

$$\hat{P}_{T=0} = \frac{1}{4}\left(1 - \hat{\tau}_1 \cdot \hat{\tau}_2\right) \quad , \quad \hat{P}_{T=1} = \frac{1}{4}\left(3 + \hat{\tau}_1 \cdot \hat{\tau}_2\right)$$

Functional form

Two-nucleon states: singlet and triplet



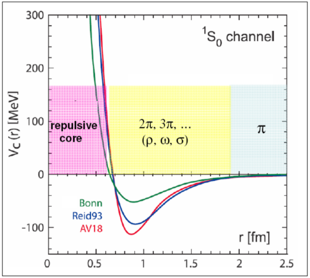
S=0, T=1: $|00;11\rangle$ $|00;10\rangle$ $|00;1-1\rangle$

S=1, T=0: $|11;00\rangle$

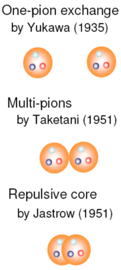
$$|SS_z;TT_z\rangle = \sum_{s_z,t_z} \mathrm{CG} |1/2, s_z; 1/2, t_z\rangle_1 \otimes |1/2, s_z; 1/2, t_z\rangle_2$$

Interactions from NN scattering

Phenomenological nuclear forces



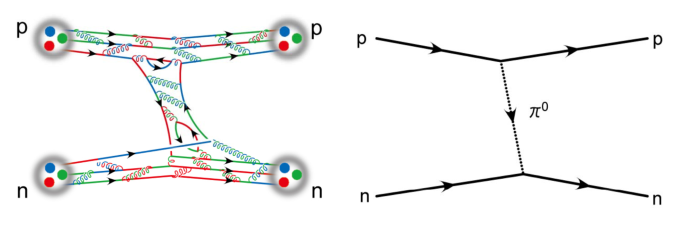
From T. Hatsuda (Oslo 2008)



Some basic features

- the interaction has a short range of about $1\mathrm{fm}$,
- within this range, it is attractive with a depth of about $40\mathrm{MeV}$ for the larger distances,
- there is strong repulsion at shorter distances $\leq 0.5\mathrm{fm}$,
- it depends both on spin and isospin of the two nucleons.

Phenomenological nuclear forces: one-pion-exchange potential



The Lagrangian density for the axial-vector type coupling of nucleon and pion fields

$$\mathcal{L}_{AV} = -\frac{g_A}{2f_\pi}\bar{\Psi}\gamma^\mu\gamma_5\boldsymbol{\tau}\Psi\cdot\partial_\mu\boldsymbol{\pi}$$

In the heavy baryon formalism (non-relativistic approximation), the AV Lagrangian becomes

$$\widehat{\mathcal{L}}_{AV} = -\frac{g_A}{2f_\pi}\bar{N}\boldsymbol{\tau}\cdot(\vec{\sigma}\cdot\vec{\nabla})\boldsymbol{\pi}N$$

The corresponding vertex in momentum space is

$$-\frac{g_A}{2f_\pi}\tau^a\vec{\sigma}\cdot\vec{q} = -\frac{g_{\pi NN}}{2M_N}\tau^a\vec{\sigma}\cdot\vec{q}$$

with $f_\pi = g_A M_N/g_{\pi NN} = 92.4$ MeV. The average nucleon mass $M_N = 938.918$ MeV, $g_A = 1.29$ and $g_{\pi NN}/2\pi = 13.67$, m_π the pion mass, and **q** the momentum transfer.

Phenomenological nuclear forces: one-pion-exchange potential

The NN interaction in momentum space

$$
\begin{aligned}
V_{1\pi}(\boldsymbol{q}) &= -\frac{g_{\pi NN}^2}{4M_N^2}\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{q})}{m_\pi^2+\mathbf{q}^2}(\hat{\boldsymbol{\tau}}_1\cdot\hat{\boldsymbol{\tau}}_2)\\
&= -\frac{g_A^2}{4f_\pi^2}\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{q})}{m_\pi^2+\mathbf{q}^2}(\hat{\boldsymbol{\tau}}_1\cdot\hat{\boldsymbol{\tau}}_2)\\
&= -\frac{g_A^2}{4f_\pi^2}(\hat{\boldsymbol{\tau}}_1\cdot\hat{\boldsymbol{\tau}}_2)\frac{1}{m_\pi^2+\mathbf{q}^2}\left(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}^{\mathbf{q}}\right)\boldsymbol{q}^2/3
\end{aligned}
$$

where the tensor operator in momentum space

$$
S_{12}^{\mathbf{q}} \equiv 3\frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})}{q^2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2.
$$

Phenomenological nuclear forces: one-pion-exchange potential

- The NN interaction in the ${}^1S_0$ channel (nn or pp):
  Total spin $S = 0$ and $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = -3$, orbital angular momentum $L = 0$, total isospin $T = 1$ and $\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2 = 1$. The tensor operator does not contribute to this channel, and the NN potential becomes,

$$
{}^{1}S_0\, V_{1\pi}(\boldsymbol{q}) = \frac{g_A^2}{4f_\pi^2} \frac{\mathbf{q}^2}{m_\pi^2 + \mathbf{q}^2} = \frac{g_A^2}{4f_\pi^2} \left(1 - \frac{m_\pi^2}{m_\pi^2 + \mathbf{q}^2}\right),
$$

  which is shown to be repulsive.

- The NN interaction in the ${}^3S_1$ channel (np):
  Total spin $S = 1$ and $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = 1$, orbital angular momentum $L = 0$, total isospin $T = 1$ and $\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2 = -3$. The NN potential becomes,

$$
{}^{3}S_1\, V_{1\pi}(\boldsymbol{q}) = \frac{g_A^2}{4f_\pi^2} \frac{\boldsymbol{q}^2}{m_\pi^2 + \mathbf{q}^2} \left(1 + S_{12}^{\mathbf{q}}\right)
$$

Phenomenological nuclear forces: one-pion-exchange potential

- The NN interaction in coordinate space is given by the Fourier transformation of the interaction in momentum space

$$
\begin{aligned}
& V_{1\pi}\left(\boldsymbol{r}_1-\boldsymbol{r}_2, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2\right) \\
& =\int d^3 q e^{i \mathbf{q} \cdot\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)} V_{1\pi}(\boldsymbol{q}) \\
& =-\frac{g_A^2}{4 f_\pi^2} \int d^3 q e^{i \mathbf{q} \cdot\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)} \frac{\left(\boldsymbol{\sigma}_1 \cdot \mathbf{q}\right)\left(\boldsymbol{\sigma}_2 \cdot \mathbf{q}\right)}{m_\pi^2+\mathbf{q}^2}\left(\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2\right) \\
& =-\frac{g_A^2}{4 f_\pi^2}\left(\hat{\boldsymbol{\sigma}}_1 \cdot \nabla_1\right)\left(\hat{\boldsymbol{\sigma}}_2 \cdot \nabla_2\right) \int d^3 q \frac{e^{i \mathbf{q} \cdot\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)}}{m_\pi^2+\mathbf{q}^2}\left(\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2\right) \\
& =-\frac{g_A^2}{4 f_\pi^2}\left(\hat{\boldsymbol{\tau}}_1 \cdot \hat{\boldsymbol{\tau}}_2\right)\left(\hat{\boldsymbol{\sigma}}_1 \cdot \nabla_1\right)\left(\hat{\boldsymbol{\sigma}}_2 \cdot \nabla_2\right) \frac{1}{4 \pi} y_\pi(r),
\end{aligned}
$$

where the function $Y_\pi(r)$ is defined as

$$
y_\pi(r)=\frac{\mathrm{e}^{-m_\pi r}}{r}, \quad \boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2
$$

Phenomenological nuclear forces: one-pion-exchange potential

With the relation,

$$\left(-\nabla^2 + m_\pi^2\right) y_\pi(r) = 4\pi\delta(\mathbf{r})$$

and rewriting

$$\begin{aligned}
&(\sigma_1 \cdot \nabla)(\sigma_2 \cdot \nabla)\, y_\pi(r) \\
&\quad = \left[(\sigma_1 \cdot \boldsymbol{\nabla})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\nabla^2\right] y_\pi(r) + \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\nabla^2 y_\pi(r)
\end{aligned}$$

$$\begin{aligned}
&\left[(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\nabla^2\right] y_\pi(r) \\
&= \left[(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{r}})(\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{r}}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\right] \times \left(m_\pi^2 + \frac{3m_\pi}{r} + \frac{3}{r^2}\right) y_\pi(r) \\
&= \frac{m_\pi^2}{3}\left(1 + \frac{3}{m_\pi r} + \frac{3}{m_\pi^2 r^2}\right) y_\pi(r) S_{12}^r
\end{aligned}$$

Phenomenological nuclear forces: one-pion-exchange potential

one finds the expression for the NN interaction in coordinate space

$$
\begin{aligned}
& V_{1\pi}(\boldsymbol{r}, \hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \hat{\boldsymbol{\tau}}_1, \hat{\boldsymbol{\tau}}_2) \\
&= -\frac{1}{3}\frac{m_\pi^2}{4\pi}\frac{g_A^2}{4f_\pi^2}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left[T_\pi(r)S_{12}^r + \left(y_\pi(r) - \frac{4\pi}{m_\pi^2}\delta(\mathbf{r})\right)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right] \\
&= \frac{1}{3}\frac{g_A^2}{4f_\pi^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\,\delta(\mathbf{r}) \\
&- \frac{1}{3}\frac{g_A^2}{4f_\pi^2}\frac{m_\pi^3}{4\pi}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left[Y_\pi(r)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + T_\pi(r)S_{12}^r\right]
\end{aligned}
$$

with

$$Y_\pi(r) = \frac{e^{-m_\pi r}}{m_\pi r} = y_\pi(r)/m_\pi$$

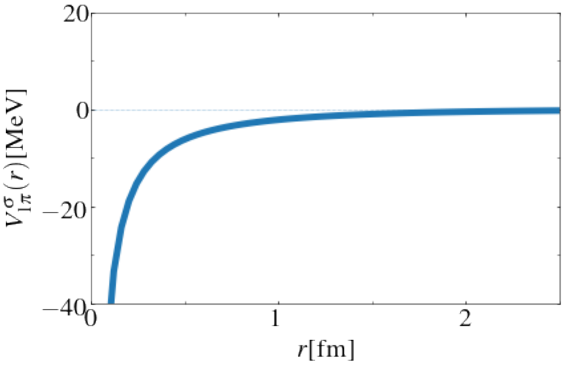
and

$$T_\pi(r) = \left(1 + \frac{3}{m_\pi r} + \frac{3}{m_\pi^2 r^2}\right) Y_\pi(r)$$

It is shown above that the $V_{1\pi}$ potential is composed of one repulsive contact term and one long-range attractive term.

Phenomenological nuclear forces: one-pion-exchange potential

$$V^{\sigma}_{1\pi}(r) \equiv -\frac{1}{3}\frac{g_A^2}{4f_\pi^2}\frac{m_\pi^3}{4\pi}\frac{e^{-m_\pi r}}{m_\pi r}.$$



Phenomenological nuclear forces: one-boson-exchange potential

The $V_{1\pi}$ potential shows some, but not all, features of a realistic NN interaction:

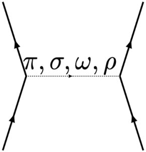
- it contains spin- and isospin-dependent parts as well as a tensor potential,
- the dominant radial dependence is of Yukawa type.

Other properties, however, show that it is not sufficient:

- there is no spin-orbit coupling and
- there is no short-range repulsion.

one-boson-exchange potential: R. Machleidt, Phys.Rev. C63 (2001) 024001

$$\mathcal{L}_{\pi^0 NN} = -g_{\pi^0}\bar{\psi} i\gamma^5 \tau_3 \psi \varphi^{(\pi^0)}$$
$$\mathcal{L}_{\pi^\pm NN} = -\sqrt{2} g_{\pi^\pm}\bar{\psi} i\gamma^5 \tau_\pm \psi \varphi^{(\pi^\pm)}$$
$$\mathcal{L}_{\sigma NN} = -g_{\sigma}\bar{\psi}\psi \varphi^{(\sigma)}$$
$$\mathcal{L}_{\omega NN} = -g_{\omega}\bar{\psi}\gamma^\mu \psi \varphi^{(\omega)}_\mu$$

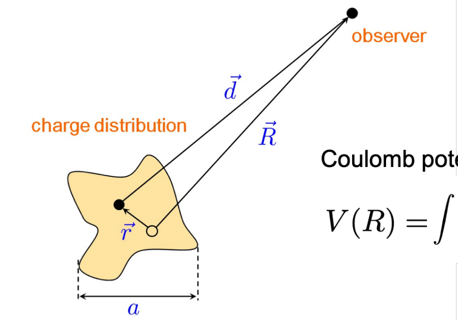


$$\mathcal{L}_{\rho NN} = -g_{\rho}\bar{\psi}\gamma^\mu \boldsymbol{\tau}\psi \cdot \boldsymbol{\varphi}^{(\rho)}_\mu - \frac{f_\rho}{4M_p}\bar{\psi}\sigma^{\mu\nu}\boldsymbol{\tau}\psi \cdot (\partial_\mu \boldsymbol{\varphi}^{(\rho)}_\nu - \partial_\nu \boldsymbol{\varphi}^{(\rho)}_\mu)$$

- the intermediate attractive is described by the exchange of the scalar meson σ.
- the short-range repulsion is described by the exchange of the vector meson ω.
- the isospin-dependence is described by the exchange of ρ.

Nuclear forces from chiral EFT

- What can we learn from the well-known Coulomb potential?



Coulomb potential:

$$V(R) = \int d^3r \, \frac{\rho(\vec{r})}{|\vec{R} - \vec{r}|}$$

How to determine the V(R) if the charge distribution is unknown?

Nuclear forces from chiral EFT

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{4\pi}{R}\sum_{L=0}\frac{1}{2L+1}(r/R)^L\sum_M Y^*_{LM}(\Omega_r)Y_{LM}(\Omega_R)$$

- Identify the relevant degree-of-freedom: r, R
- A quantity much smaller than 1: $r/R << 1$
- order-by-order convergence: $(r/R)^L$

- The LO ($L=0$) term:

$$V^{LO}(R) = \int d^3r\rho(\vec{r})\frac{4\pi}{R}Y_{00}(\Omega_r)Y^*_{00}(\Omega_R) = \frac{1}{R}\int d^3r\rho(\vec{r})$$

- The NLO ($L=1$) term:

$$V^{NLO}(R) = \int d^3r\rho(\vec{r})\frac{4\pi}{R}\sum_{M=-1}^{1}\frac{1}{3}(r/R)Y^*_{1M}(\Omega_r)Y_{1M}(\Omega_R) = \frac{1}{R^3}\int d^3r\rho(\vec{r})\vec{r}\cdot\vec{R}$$

Nuclear forces from chiral EFT

- The NNLO ($L=2$) term:

$$V^{N^2LO}(R) = \int d^3r\rho(\vec{r})\frac{4\pi}{R}\sum_{M=-2}^{2}\frac{1}{5}(r/R)^2 Y^*_{2M}(\Omega_r)Y_{2M}(\Omega_R)$$

$$= \frac{1}{R^5}\frac{1}{5}\sum_{M=-2}^{2}\vec{R}^2 Y_{2M}(\Omega_R)\int d^3r\rho(\vec{r})\vec{r}^2 Y^*_{2M}(\Omega_r)$$

Put them together,

$$\int d^3r\,\frac{\rho(\vec{r})}{|\vec{R}-\vec{r}|} = \underbrace{\frac{q}{R}}_{\text{LO}} + \underbrace{\frac{1}{R^3}\sum_i R_i P_i}_{\text{NLO}} + \underbrace{\frac{1}{6R^5}\sum_{ij}(3R_iR_j-\delta_{ij}R^2)Q_{ij}}_{\text{N}^2\text{LO}} + \ldots$$

The result is systematically improvable

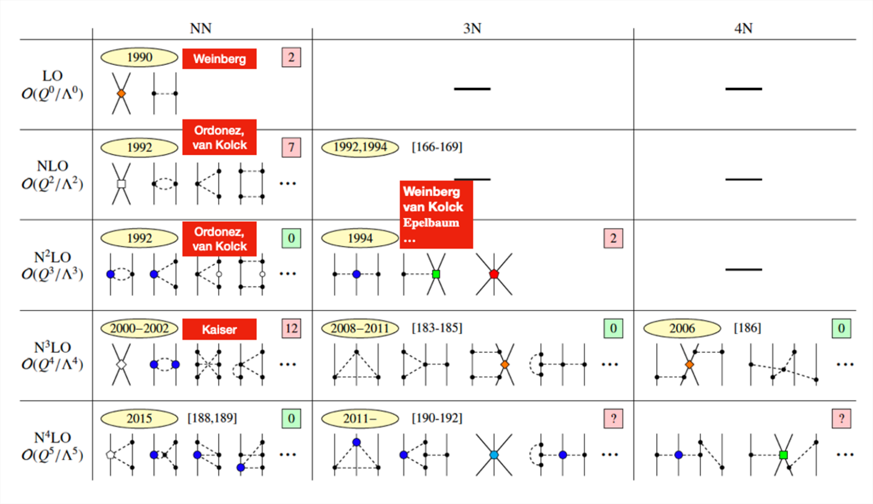
$$q=\int d^3r\,\rho(\vec{r}),\qquad P_i=\int d^3r\,\rho(\vec{r})\,r_i,\qquad Q_{ij}=\int d^3r\,\rho(\vec{r})(3r_ir_j-\delta_{ij}r^2)$$

Nuclear forces from chiral EFT

For the NN interaction

- Identify the relevant degree-of-freedom: Q, m_π, Λ_χ
- A quantity much smaller than 1: $(Q, m_\pi)/\Lambda_\chi \approx 0.14 << 1$, where Q is the kinetic energy of nucleons, $\Lambda_\chi \sim 1$ GeV – chiral symmetry breaking scale.

Nuclear forces from chiral EFT



K. Hebeler, Phys. Rep. 890, 1 (2020)

Effective interactions

Nuclear effective interactions

- The realistic NN interaction (in free space) has a “hard" core (large repulsive at the short distance).
- The convergence of many-body approaches using the realistic NN interaction is very slow.
- The NN interaction in atomic nuclei is modified by many-body correlations and thus an effective NN interaction is more suitable for nuclear structure calculations.

The most popular effective interactions

- The Skyrme force
- The Gogny force
- The effective Lagrangian of relativistic mean-field (RMF) theory

Nuclear effective interactions: The Skyrme force

The most used effective interaction in the Hartree-Fock (HF) calculation is the Skyrme force:

$$\hat{V} = \sum_{i<j} \hat{v}_{ij}^{(2)} + \sum_{i<j<k} \hat{v}_{ijk}^{(3)}$$

The two-body interaction contains momentum dependence as well as spin-exchange contributions and a spin-orbit force:

$$\begin{aligned}
\hat{v}_{ij}^{(2)} =& t_0 \left(1 + x_0 \hat{P}_\sigma\right) \delta\left(r_i - r_j\right) \\
&+ \frac{1}{2} t_1 \left(\delta\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \hat{k}^2 + \hat{k}'^2 \delta\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)\right) t_2 \hat{\boldsymbol{k}}' \cdot \delta\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \hat{\boldsymbol{k}} \\
&+ \mathrm{i} W_0 \left(\hat{\boldsymbol{\sigma}}_i + \hat{\boldsymbol{\sigma}}_j\right) \cdot \hat{\boldsymbol{k}}' \times \delta\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \hat{\boldsymbol{k}}
\end{aligned}$$

Here, instead of the operator of relative momentum the related expressions

$$\hat{k} = \frac{1}{2\mathrm{i}} \left(\nabla_i - \nabla_j\right) \quad , \quad \hat{\boldsymbol{k}}' = -\frac{1}{2\mathrm{i}} \left(\nabla_i - \nabla_j\right)$$

are used with the additional convention that $\hat{k}'$ acts on the wave function to its left. The three-body interaction is a purely local potential

$$\hat{v}_{ijk}^{(3)} = t_3 \delta\left(r_i - r_j\right) \delta\left(r_j - r_k\right)$$

The Skyrme forces contain six parameters t_0, t_1, t_2, t_3, x_0, and W_0, which are fitted to reproduce properties of finite nuclei within a Hartree-Fock calculation.

Nuclear effective interactions: The Gogny force

The finite-range Gogny force

$$
\begin{aligned}
V_{NN,12} = & \sum_{i=1}^{2} e^{-(\vec{r}_1-\vec{r}_2)^2/\mu_i^2}\left(W_i + B_i P^\sigma - H_i P^\tau - M_i P^\sigma P^\tau\right) \\
& + t_3 \delta\left(\vec{r}_1 - \vec{r}_2\right)\left(1 + x_0 P^\sigma\right)\left[\rho\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)\right]^\alpha \\
& + iW_0 \delta\left(\vec{r}_1 - \vec{r}_2\right)\left(\vec{\sigma}_1 + \vec{\sigma}_2\right)\cdot \vec{k}' \times \vec{k}
\end{aligned}
$$

where $P^\sigma = \frac{1}{2}\left(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2\right)$ and $P^\tau = \frac{1}{2}\left(1 + \vec{\tau}_1 \cdot \vec{\tau}_2\right)$ are the spin- and isospin-exchange operators.

Table: The D1S parameters for the Gogny force [J. Berger, M. Girod, and D. Gogny, Comp. Phys. Comm. 63, 365 (1991)]

	μ_i(fm)	W_i(MeV)	B_i(MeV)	H_i(MeV)	M_i(MeV)	W_0(MeV)	t_3(MeV)	x_0	α
$i=1$	0.7	−1720.30	1300.00	−1813.53	1397.60				
$i=2$	1.2	103.64	−163.48	162.81	−223.93	130	1390.60	1	1/3

Nuclear effective interactions: The RMF theory

The Lagrangian density of the RMF theory:

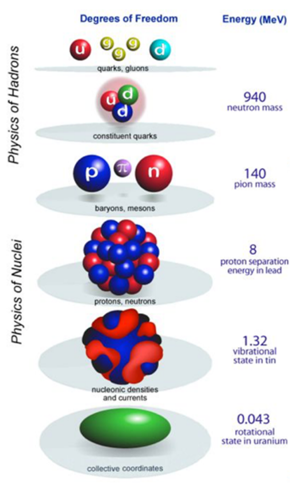
$$
\begin{aligned}
\mathcal{L} = & \bar{\psi}\left[i\gamma^\mu\partial_\mu - M - g_\sigma\sigma - g_\omega\gamma^\mu\omega_\mu - g_\rho\gamma^\mu\vec{\tau}\cdot\vec{\rho}_\mu - e\gamma^\mu A_\mu\frac{1-\tau_3}{2}\right]\psi \\
& + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{3}g_2\sigma^3 - \frac{1}{4}g_3\sigma^4 \\
& - \frac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu + \frac{1}{4}c_3\left(\omega_\mu\omega^\mu\right)^2 \\
& - \frac{1}{4}\vec{R}_{\mu\nu}\vec{R}^{\mu\nu} + \frac{1}{2}m_\rho^2\vec{\rho}_\mu\cdot\vec{\rho}^\mu \\
& - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}
\end{aligned}
$$

in which the field tensors for the vector mesons and the photon are respectively defined as,

$$
\begin{cases}
\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu \\
\vec{R}_{\mu\nu} = \partial_\mu\vec{\rho}_\nu - \partial_\nu\vec{\rho}_\mu \\
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\end{cases}
$$

Summary

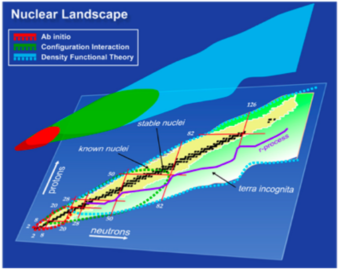
The NN interaction: an essential ingredient of nuclear theory



multi-faceted nuclei

- **Nuclear many-body calculations (challenge)**

  ✓ Ab initio methods
  ✓ Configuration-interaction shell-models
  ✓ Nuclear energy density functionals
  ✓ Collective models
  ✓ ...



The Frontiers of Nuclear Science: A Long-Range Plan, 2007.

Appendix

The Gell-Mann matrices

A set of eight linearly independent 3×3 traceless Hermitian matrices used in the study of the strong interaction in particle physics. They span the Lie algebra of the SU(3) group in the defining representation.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The gluons

The "eight types" or "eight colors" of gluons:

$$
\begin{array}{ll}
(r\bar{b} + b\bar{r})/\sqrt{2} & -i(r\bar{b} - b\bar{r})/\sqrt{2} \\
(r\bar{g} + g\bar{r})/\sqrt{2} & -i(r\bar{g} - g\bar{r})/\sqrt{2} \\
(b\bar{g} + g\bar{b})/\sqrt{2} & -i(b\bar{g} - g\bar{b})/\sqrt{2} \\
(r\bar{r} - b\bar{b})/\sqrt{2} & (r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}
\end{array}
$$